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A Form of the Borel–Cantelli Lemma

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INTRODUCTION

In this paper we prove a strong version of the Borel–Cantelli lemma. Our proof is based on a convergence theorem in martingale theory and a local convergence theorem for sequences of positive random variables [5]. We then apply our result (Theorem 2) to deduce both a form of the Borel–Cantelli lemma due to Dubins and Freedman [1], and a more recent result due to Hill [2]. We also prove a strong law of large numbers for sequences of positive random variables (Theorem 4), and as an application we prove a convergence theorem on infinite series that is a generalization of a theorem of Dini [2, 3].

1. PRELIMINARIES

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_n, n \geqslant 1)$ an increasing sequence of sub-σ-fields of $\mathscr{F}$. A sequence $(X_n, n \geqslant 1)$ of random variables is said to be adapted if for each $n \geqslant 1$, X_n is $\mathscr{F}_n$-measurable. For $p \in [1, +\infty)$ we denote by $L^p(\Omega, \mathscr{F}, P)$ the space of random variables of pth integrable power. For an adapted sequence of integrable random variables $(X_n, n \geqslant 1)$ we define $(S_n, n \geqslant 1)$ by,

$$S_n = \sum_{i=1}^{n} X_i.$$

We define $(s_n, n \geqslant 1)$ by,

$$s_1 = 0$$

$$s_{n+1} = \sum_{i=1}^{n} E(X_{i+1} \mid \mathscr{F}_i), \qquad n \geqslant 1.$$

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If moreover $(X_n, n \geqslant 1)$ is in $L^p(\Omega, \mathscr{F}, P)$ for some $p \in [1, +\infty)$ we define $(A_n^{(p)}, n \geqslant 1)$ by

$$A_1^{(p)} = 0$$

$$A_{n+1}^{(p)} = \sum_{i=1}^{n} E(|X_{i+1}|^p \mid \mathscr{F}_i), \qquad n \geqslant 1.$$

2. Convergence Results

We start out with a lemma.

Lemma 1. *Let $(X_n, n \geqslant 1)$ be an adapted sequence of positive random variables in $L^p(\Omega, \mathscr{F}, P)$ for some $p \in [1, +\infty)$. Let $f\colon [0, \infty) \to \mathbb{R}$ be a strictly positive and increasing function such that $\int_0^{+\infty} f^{-p}(x)\, dx < \infty$. Then,*

$$\sum_{i=1}^{+\infty} f^{-p}(A_{i+1}^{(p)})\, E(|X_{i+1} - E(X_{i+1} \mid \mathscr{F}_i)|^p \mid \mathscr{F}_i) < \infty \qquad \text{a.s.}$$

Proof. For $i \geqslant 1$, $|X_{i+1} - E(X_{i+1} \mid \mathscr{F}_i)|^p \leqslant 2^p(|X_{i+1}|^p + |E(X_{i+1} \mid \mathscr{F}_i)|^p)$. Therefore by Jensen's inequality, by the definition of $A_i^{(p)}$ and by the fact that $X_i \geqslant 0$ for each $i \geqslant 1$, we have

$$\begin{aligned} E(|X_{i+1} - E(X_{i+1} \mid \mathscr{F}_i)|^p \mid \mathscr{F}_i) &\leqslant 2^{p+1} E(X_{i+1}^p \mid \mathscr{F}_i) \\ &= 2^{p+1}(A_{i+1}^{(p)} - A_i^{(p)}). \end{aligned}$$

Hence it is enough to show that

$$\sum_{i=1}^{+\infty} f^{-p}(A_{i+1}^{(p)})(A_{i+1}^{(p)} - A_i^{(p)}) < \infty \qquad \text{a.s.}$$

But the fact that f is increasing implies that

$$\begin{aligned} \sum_{i=1}^{+\infty} f^{-p}(A_{i+1}^{(p)})(A_{i+1}^{(p)} - A_i^{(p)}) &\leqslant \sum_{i=1}^{+\infty} \int_{A_i^{(p)}}^{A_{i+1}^{(p)}} f^{-p}(x)\, dx \\ &\leqslant \int_0^{+\infty} f^{-p}(x)\, dx < \infty. \end{aligned}$$

We now prove the main theorem.

Theorem 2. *Let $(X_n, n \geqslant 1)$ be an adapted sequence of random variables such that $0 \leqslant X_n \leqslant 1$ for each $n \geqslant 1$. Let $f\colon [0, \infty) \to \mathbb{R}$ be a strictly* *positive and increasing function such that* $\int_0^{+\infty} f^{-p}(x)\,dx < \infty$ *for some* $p \in [1, 2]$. *Then*

$$\lim_{n \to +\infty} \frac{S_n - s_n}{f(s_n)} < \infty \qquad \text{a.s.}$$

The limit is equal to 0 on the set $[\lim_{n \to +\infty} s_n = +\infty]$.

Proof. We first prove that $\lim_{n \to +\infty} [(S_n - s_n)/f(s_n)] = 0$ a.s. on the set $[\lim_{n \to +\infty} s_n = +\infty]$. Let $M_n = S_n - s_n$ for each $n \geqslant 1$. Since $M_{n+1} - M_n = X_{n+1} - E(X_{n+1} \mid \mathscr{F}_n)$ $(M_n, n \geqslant 1)$ is a martingale. Moreover, since $p \geqslant 1$ and $0 \leqslant X_i \leqslant 1$ for each $i \geqslant 1$,

$$A_{i+1}^{(p)} = \sum_{j=1}^{i} E(X_{j+1}^p \mid \mathscr{F}_j) \leqslant \sum_{j=1}^{i} E(X_{j+1} \mid \mathscr{F}_j) = s_{i+1}.$$

Hence, $f^{-p}(s_{i+1}) \leqslant f^{-p}(A_{i+1}^{(p)})$ and by Lemma 1,

$$\sum_{i=1}^{+\infty} f^{-p}(s_{i+1})\, E(|M_{i+1} - M_i|^p \mid \mathscr{F}_i) < \infty \qquad \text{a.s.}$$

Now let $W_1 = 0$, and for $n \geqslant 1$ $W_{n+1} = \sum_{i=1}^{n} f^{-1}(s_{i+1})(M_{i+1} - M_i)$. For every $n \geqslant 1$, $f(s_{n+1})$ is $\mathscr{F}_n$-measurable. Hence $(W_n, n \geqslant 1)$ is a martingale and $\sum_{i=1}^{+\infty} E(|W_{i+1} - W_i|^p \mid \mathscr{F}_i) < \infty$ a.s. Therefore by Corollary 2.8.5 [4, p. 67], W_n converges a.s. Since $\lim_{x \to +\infty} f(x) = +\infty$ and f is increasing, $[\lim_{n \to +\infty} f(s_n) = +\infty] = [\lim_{n \to +\infty} s_n = +\infty]$. Hence by Kronecker's lemma $\lim_{n \to +\infty} [M_n/f(s_n)] = 0$ a.s. on $[\lim_{n \to +\infty} s_n = +\infty]$. To prove the convergence on the set $[\lim_{n \to +\infty} s_n < \infty]$ we apply Theorem 2 in [5]. It says that on the set $[\sup_{n \geqslant 1} s_n < \infty]$, S_n converges a.s. if and only if s_n converges a.s. Therefore S_n converges a.s. on $[\lim_{n \to +\infty} s_n < \infty]$. Moreover the fact that f is increasing implies that $f(s_n)$ converges on $[\lim_{n \to +\infty} s_n < +\infty]$. Hence $(S_n - s_n)/f(s_n)$ converges on $[\lim_{n \to +\infty} s_n < \infty]$. Q.E.D.

Next we use Theorem 2 to deduce the versions of the Borel–Cantelli lemma given in [1, 2]. The sequence $(X_n, n \geqslant 1)$ will be as in Theorem 2, and for $k \geqslant 1$ we let $\log_k x$ be the kth iterated logarithm of x (e.g., $\log_3 x = \log\log\log x$).

COROLLARY 3. (i) *For every integer* $k \geqslant 1$ *and for every* $p \in (1, 2]$,

$$\lim_{n \to +\infty} \frac{S_n - s_n}{(s_n \log_1(s_n) \cdots \log_{k-1}(s_n) \log_k^p(s_n))^{1/p}} = 0$$

$$\text{a.s.} \quad \mathit{on} \left[\lim_{n \to +\infty} s_n = +\infty \right].$$

(ii) *If f: $[0, \infty) \to \mathbb{R}$ is a strictly positive and increasing function such that* $\lim_{x\to+\infty} [x/f(x)] = a < \infty$ *then $S_n/f(s_n)$ converges a.s. to a finite limit. The limit is equal to a on the set* $[\lim_{n\to+\infty} s_n = +\infty]$.

Proof. (i) For $k \geqslant 1$ we define f_k: $[a_k, +\infty) \to \mathbb{R}$ by $f_k(x) = (x \log_1 x \cdots \log_{k-1} x \log_k^p x)^{1/p}$ where a_k is taken to be large enough so that f_k is well defined, strictly positive and increasing. Let f_k^*: $[0, \infty) \to \mathbb{R}$ be the extension of f that is identically equal to $f_k(a_k)$ on the interval $[0, a_k)$; f_k^* is strictly positive and increasing. Moreover,

$$\int_{a_k}^{+\infty} f_k^{-p}(x)\, dx = ((p-1) \log_k^{p-1}(a_k))^{-1} < \infty.$$

Therefore $\int_0^{+\infty} (f_k^*(x))^{-p}\, dx < \infty$. Hence, by Theorem 2,

$$\lim_{n\to+\infty} \frac{S_n - s_n}{f_k^*(s_n)} = \lim_{n\to+\infty} \frac{S_n - s_n}{f_k(s_n)} = 0$$

$$\text{a.s.} \quad \text{on} \left[\lim_{n\to+\infty} s_n = +\infty \right].$$

(ii) The fact that $\lim_{x\to+\infty} [x/f(x)]$ exists and is finite implies that $\int_0^{+\infty} f^{-2}(x)\, dx < \infty$. Therefore, by Theorem 2, $(S_n - s_n)/f(s_n)$ converges a.s. We have a.s. on the set $[\lim_{n\to+\infty} s_n = +\infty]$,

$$\lim_{n\to+\infty} \frac{S_n}{f(s_n)} = \lim_{n\to+\infty} \left(\frac{S_n - s_n}{f(s_n)} + \frac{s_n}{f(s_n)} \right)$$

$$= \lim_{x\to+\infty} \frac{x}{f(x)} = a.$$

On the set $[\lim_{n\to+\infty} s_n < \infty]$, $f(s_n)$ converges because f is increasing, therefore $S_n/f(s_n)$ converges a.s. Q.E.D.

Remarks. (1) Part (ii) of Corollary 3 was proved by Dubins and Freedman [1] for $f(x) = x + EX_1$ with $EX_1 > 0$. Part (i) is due to Hill [2].

(2) It is an easy exercise to generalize Theorem 2 to an adapted sequence of nonnegative uniformly bounded random variables.

(3) Suppose that f satisfies the following condition: for any two sequences of real numbers $(a_n, n \geqslant 1)$ and $(b_n, n \geqslant 1)$ such that $\lim_{n\to+\infty} (a_n/b_n) = 1$, $\lim_{n\to+\infty} [f(a_n)/f(b_n)] < \infty$. Then under the assumptions of Theorem 2 (and $EX_1 > 0$), $(S_n - s_n)/f(S_n)$ converges a.s. to a finite limit. The limit is equal to 0 on $[\lim_{n\to+\infty} s_n = +\infty]$. The convergence on $[\lim_{n\to+\infty} S_n = +\infty]$ follows from Theorem 2 and Remark 1 above. On $[\lim_{n\to+\infty} s_n < \infty]$ we use the same argument as in the proof of Theorem 2.

The functions $(f_k, k \geqslant 1)$ in the proof of Corollary 3 satisfy the condition above.

Next we prove a strong law of large numbers for sequences of positive random variables. As an application we prove a result on infinite series that is a simultaneous generalization of a theorem of Dini [3, p. 290], and Lemma 1 in [2].

THEOREM 4. *Let $(X_n, n \geqslant 1)$ be an adapted sequence of positive integrable random variables. Let $f\colon [0, +\infty) \to \mathbb{R}$ be a strictly positive and increasing function such that $\int_0^{+\infty} f^{-1}(x)\,dx < \infty$. Then,*

(i) $\sum_{n=1}^{+\infty} [X_n/f(s_n)] < \infty$ a.s.

(ii) $\lim_{n\to+\infty} [S_n/f(s_n)] = 0$ a.s. *on* $[\lim_{n\to+\infty} s_n = +\infty]$.

Proof. (i) For each $n \geqslant 1$, let $W_n = \sum_{i=1}^{n} f^{-1}(s_i) X_i$; $(W_n, n \geqslant 1)$ is adapted and positive. Moreover, since $f(s_{n+1})$ is $\mathcal{F}_n$-measurable and f is increasing,

$$
\begin{aligned}
0 \leqslant E(W_{n+1} - W_n \mid \mathcal{F}_n) &= E(f^{-1}(s_{n+1}) X_{n+1} \mid \mathcal{F}_n)\\
&= f^{-1}(s_{n+1})(s_{n+1} - s_n)\\
&\leqslant \int_{s_n}^{s_{n+1}} f^{-1}(x)\,dx.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{n=1}^{+\infty} E(W_{n+1} - W_n \mid \mathcal{F}_n) &\leqslant \sum_{n=1}^{+\infty} \int_{s_n}^{s_{n+1}} f^{-1}(x)\,dx\\
&\leqslant \int_0^{\infty} f^{-1}(x)\,dx.
\end{aligned}
$$

Therefore

$$
\sum_{n=1}^{+\infty} E(W_{n+1} - W_n \mid \mathcal{F}_n) < \infty \qquad \text{a.s.}
$$

Applying Theorem 2 in [5] we conclude that W_n converges a.s.

(ii) Since $\lim_{x\to+\infty} f(x) = +\infty$ and f is increasing,

$$
\left[\lim_{n\to+\infty} f(s_n) = +\infty\right] = \left[\lim_{n\to+\infty} s_n = +\infty\right].
$$

The conclusion follows by using (i) and Kronecker's lemma. Q.E.D.

COROLLARY 5. *Let* $(d_n, n \geqslant 1)$ *be a sequence of positive numbers with* $D_n = \sum_{i=1}^{n} d_i$. *Let* $f\colon [0, \infty) \to \mathbb{R}$ *be a strictly positive and increasing function such that* $\int_0^{+\infty} f^{-1}(x)\,dx < \infty$. *Then*

$$\sum_{n=1}^{+\infty} \frac{d_n}{f(D_n)} < \infty.$$

Proof. Without loss of generality we may assume $d_1 = 0$. Let $X_n = d_n$ for each $n \geqslant 1$. Therefore $S_n = s_n = D_n$. The conclusion follows by applying Theorem 4(i) to the sequence $(X_n, n \geqslant 1)$. Q.E.D.

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